Character rings in algebraic topology

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1 Introduction

Let $R(G)$ be the complex representation ring of a finite group G. The assignment

$$G \mapsto R(G)$$

is a Mackey functor, i.e., a functor from groups to rings endowed with induction (transfer) maps. Typically one studies this functor via characters. One lets $C(G)$ be the ring of complex-valued functions on G invariant under conjugation, and then one defines a natural map of rings

$$\chi: R(G) \mapsto C(G)$$

by $\chi(M)(g) = \operatorname{trace}\{g: M \to M\}$. There is a simple formula defining induction between character rings making χ a map of Mackey functors.

Why is χ so important? There are three reasons:

(i) χ is injective,

(ii) χ is as surjective as possible: $R(G) \subset C(G)$ is a maximal $\mathbf{Z}$-lattice, so that $R(G) \otimes_{\mathbf{Z}} \mathbf{C} \simeq C(G)$, and

(iii) $C(G)$ is very concrete and easy to work with.

Analogues of $R(G)$ are well known to topologists: given a multiplicative cohomology theory E^*, the assignment

$$G \mapsto E^*(BG)$$

is a Mackey functor.

In this paper, I discuss some natural generalizations of the classical character ring that can be used to detect $E^*(BG)$, for various E^*, in much the same way that $C(G)$ detects $R(G)$. Two recent projects fit into this picture: my joint work with

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M. Hopkins and D. Ravenel on v_n-periodic complex oriented theories [HKR], and the reinterpretation and extension of Quillen's work on $H^*(BG)$ by J. Lannes and L. Schwartz and others (see e.g., [HLS]).

To explain how one generalizes $C(G)$, observe that $G \simeq \mathrm{Hom}(\mathbf{Z}, G)$. Letting G act trivially on $\mathbf{C}$ and by conjugation on $\mathrm{Hom}(\mathbf{Z}, G)$, we have a natural isomorphism of $\mathbf{C}$-algebras

$$C(G) \simeq \mathrm{Map}_G(\mathrm{Hom}(\mathbf{Z}, G), \mathbf{C}).$$

We will replace $\mathbf{Z}$ by other groups Γ and $\mathbf{C}$ by other fields $\mathbf{F}$. In the examples, the choice of Γ seems to be vitally linked to the geometry underlying the cohomology theory E^*.

We discuss the organization of the paper.

In Section 2 and Section 3 we discuss formal properties of our generalized character rings. In particular, they are Mackey functors, and one is naturally led to inverse limits over categories of detecting subgroups.

The next two sections have parallel descriptions of the two projects referred to above: Section 4 reviews the $H^*(BG)$ results, while Section 5 describes the characters for complex oriented theories. The Lannes-Schwartz work leads to a purely group theoretic criterion guaranteeing, for a finite p-group G, that $H^*(BG; \mathbf{Z}/p)$ has an infinite-dimensional A-module summand in the kernel of Quillen's map. The criterion is satisfied when G is the quaternionic group of order 8. Reasoning in an analogous manner about the nth Morava K-theory $K(n)^*(BG)$, we give a group theoretic criterion that would imply that $K(n)^*(BG)$ is *not* concentrated in even degrees,as has been conjectured. Fortunately, or unfortunately, we have yet to find a group satisfying our condition — indeed, there are theorems in group theory hinting that it cannot be done.

In Section 6, we briefly describe how our character rings extend to equivariant cohomology theories. As a simple example, we show how classical character theory extends to a computation of $K_G(X) \otimes \mathbf{C}$ in non-equivariant terms.

The last section touches on the possibility of assembling character rings into simplicial objects.

We would like to thank Mike Hopkins for providing the proof of Lemma 5.6 and Leonard Scott for bringing our attention to the theorem of John Thompson used in proving Proposition 5.7.

2 Character rings: definitions and examples

We begin by defining our rings of class functions. Given two topological groups Γ and G, we let $\mathrm{Hom}(\Gamma, G)$ denote the space of continuous homomorphisms. This becomes a (left) G-space by letting G act on itself by conjugation. In the examples we consider, G will be a finite group and $\mathrm{Hom}(\Gamma, G)$ will be finite and discrete. Let F be a field (possibly graded).

2.1 Definition. $C_{\Gamma,\mathsf{F}}(G) = \mathrm{Map}_G(\mathrm{Hom}(\Gamma, G), \mathsf{F})$.

Thus an element of $C_{\Gamma,\mathsf{F}}(G)$ is a function

$$f\colon \mathrm{Hom}(\Gamma, G) \longrightarrow \mathsf{F},$$

constant on G-orbits. $C_{\Gamma,\mathsf{F}}(G)$ is an F-algebra using pointwise multiplication and addition of functions. It is clearly a contravariant functor of G, and a covariant functor of Γ and F.

To show that $C_{\Gamma,\mathsf{F}}(G)$ is a Mackey functor, we need to define induction. The classical formula [S, page 30] generalizes.

2.2 Definition. *Let H be a subgroup of a finite group G. Define*

$$ind_H^G\colon C_{\Gamma,\mathsf{F}}(H) \longrightarrow C_{\Gamma,\mathsf{F}}(G)$$

by the formula

$$ind_H^G(f)(\alpha) = \sum_{gH \in (G/H)^{\mathrm{Im}(\alpha)}} f(g \cdot \alpha),$$

where $f\colon Hom(\Gamma, H) \to \mathsf{F}$, $\alpha\colon \Gamma \to G$, and $g \cdot \alpha$ denotes α composed with conjugation by g.

It is straightforward to check that this is well-defined.

2.3 Proposition. *With this structure, the assignment $G \mapsto C_{\Gamma,\mathsf{F}}(G)$ is a Mackey functor.*

Proof. We need only check that the double coset formula holds. Let K and H be subgroups of G and $\mathrm{res}_G^K\colon C_{\Gamma,\mathsf{F}}(G) \to C_{\Gamma,\mathsf{F}}(K)$ the restriction. Letting K_g denote $K \cap gHg^{-1}$, there is an isomorphism of left K-sets

$$\coprod_{KgH} K/K_g = G/H,$$

given by sending kK_g to kgH. Thus

$$\begin{aligned}
\mathrm{res}_G^K(\mathrm{ind}_H^G(f))(\alpha) &= \sum_{gH \in (G/H)^{\mathrm{Im}(\alpha)}} f(g \cdot \alpha) \\
&= \sum_{KgH} \sum_{kK_g \in (K/K_g)^{\mathrm{Im}(\alpha)}} f(kg \cdot \alpha) \\
&= \sum_{KgH} \mathrm{ind}_{K_g}^K(f)(g \cdot \alpha),
\end{aligned}$$

as needed. ∎

2.4 Examples. (1) $\Gamma = \mathbf{Z}$. Then $\mathrm{Hom}(\mathbf{Z}, G) = G$, and $C_{\mathbf{Z},\mathbf{C}}(G)$ is the usual character ring detecting $R(G)$.

(2) $\Gamma = \mathbf{Z}[\frac{1}{p}]$. Then $\mathrm{Hom}(\mathbf{Z}[\frac{1}{p}], G) = G_{\mathrm{reg}}$, the "$p$-regular" elements in G of order prime to p. Note that the inclusion $G_{\mathrm{reg}} \subset G$ is induced by $\mathbf{Z} \to \mathbf{Z}[\frac{1}{p}]$. The ring $C_{\mathbf{Z}[\frac{1}{p}],\mathbf{F}}(G)$ arises in Brauer character theory. Let A be a p-adic ring of integers with quotient field K and residue field k. Assume that A contains the $|G|$th roots of 1. Let $R_K(G)$ (respectively $R_k(G)$) be the Grothendieck ring of $K[G]$ ($k[G]$) modules. There is a commutative diagram of ring homomorphisms:

$$\begin{array}{ccc} R_K(G) & \xrightarrow{\chi_K} & C_{\mathbf{Z},K}(G) \\ {\scriptstyle d}\downarrow & & \downarrow \\ R_k(G) & \xrightarrow{\chi_k} & C_{\mathbf{Z}[\frac{1}{p}],K}(G) \end{array}$$

where χ_K is the usual character map, χ_k is the Brauer character map, and d is the "decomposition" map [S]. Both χ_K and χ_k are inclusions of maximal $\mathbf{Z}$-lattices, and d is surjective.

(3) $\Gamma = \mathbf{Z}_p$, the p-adic integers. $\mathrm{Hom}(\mathbf{Z}_p, G)$ is the set of "p-unipotent" elements in G, the elements of order a power of p. Where does $C_{\mathbf{Z}_p,\mathbf{F}}(G)$ occur naturally as a character ring? Let $K(BG)_p$ denote the p-adic completion of the complex K-theory of BG. Atiyah's theorem [A] and the methods of [K1] show that there is a unique continuous extension of χ in the diagram

$$\begin{array}{ccc} R(G) & \xrightarrow{\chi} & C_{\mathbf{Z},\overline{\mathbf{Q}}_p}(G) \\ \downarrow & & \downarrow \\ K(BG)_p & \xrightarrow{\chi_p} & C_{\mathbf{Z}_p,\overline{\mathbf{Q}}_p}(G) \end{array}$$

and that χ_p is the embedding of a maximal $\mathbf{Z}_p$-lattice.

(4) $\Gamma = \mathbf{Z}^n$. Then $\mathrm{Hom}(\mathbf{Z}^n, G)$ is the set of n-tuples of commuting elements in G. The case $n = 2$ seems to occur in "elliptic" settings — see e.g., work by S. Norton on "montrous moonshine" [N]. Note that if C is a complex elliptic curve, $\pi_1(C) = \mathbf{Z} \times \mathbf{Z}$. See Section 6 for one Mackey functor detected by $C_{\mathbf{Z}^n,\mathbf{F}}(G)$.

(5) $\Gamma = (\mathbf{Z}_p)^n$. This is the p-adic version of (4). $\mathrm{Hom}(\mathbf{Z}_p^n, G)$ is the set of n-tuples of elements in G generating an abelian p-group. For "naturally occurring" instances of $C_{\mathbf{Z}_p^n,\overline{\mathbf{Q}}_p}(G)$ and $C_{\mathbf{Z}_p^n,\overline{\mathbf{F}}_p}(G)$, see Section 5; these rings try to detect $E^*(BG)$, where E^* is a p-local v_n-periodic oriented theory.

(6) $\Gamma = (\mathbf{Z}/p)^n$. Then $\mathrm{Hom}((\mathbf{Z}/p)^n, G)$ is the set of n-tuples generating an elementary abelian p-group. Quillen's theorem essentially says that, if $n \geq \mathrm{rank}(G)$, then $C_{(\mathbf{Z}/p)^n,\mathbf{Z}/p}(G)$ detects $H^*(BG; \mathbf{Z}/p)$ up to F-isomorphism. See Section 4.

(7) $\Gamma = F_n$, the free group on n generators. As a reminder that Γ might not be abelian, we point out that $G^n = \mathrm{Hom}(F_n, G)$. See Section 7.

3 Detecting families and counting orbits

This section exploits the obvious observation that the image of a homomorphism $\Gamma \to G$ is simultaneously a quotient group of Γ and a subgroup of G. Thus, for example, it is intuitively clear that if Γ is abelian, $\mathrm{Hom}(\Gamma, G)$ should only depend on $\mathrm{Hom}(\Gamma, A)$ where A runs through the abelian subgroups of G.

To be more formal, let $\mathcal{S}(G)$ be the category whose objects are the subgroups of G, and whose morphisms are generated by inclusions $H < K$ and conjugation by elements $g \in G$, $c_g\colon H \to gHg^{-1}$. Let $\mathcal{J}(G) \subset \mathcal{S}(G)$ just have inclusions for morphisms.

3.1 Definition. *Let $\Gamma(G)$ be the full subcategory of $\mathcal{S}(G)$ with objects those subgroups of G occurring as quotient groups of Γ. Let $\mathcal{J}\Gamma(G) = \Gamma(G) \cap \mathcal{J}(G)$.*

The observation at the beginning of this section can be more precisely stated as:

3.2 Proposition. *The natural map*

$$C_{\Gamma,\mathsf{F}}(G) \longrightarrow \lim_{H \in \Gamma(G)} C_{\Gamma,\mathsf{F}}(H)$$

is an isomorphism. The same is true if $\Gamma(G)$ is replaced by any full subcategory of $\mathcal{S}(\mathcal{G})$ containing $\Gamma(G)$.

Proof. There is an evident bijection of G-sets

$$\operatorname*{colim}_{H \in \mathcal{J}\Gamma(G)} \mathrm{Hom}(\Gamma, H) \longrightarrow \mathrm{Hom}(\Gamma, G).$$

Taking G-invariant maps into F yields the isomorphism, noting that, for any contravariant functor F,

$$\lim_{H \in \Gamma(G)} F(H) = \Big(\lim_{H \in \mathcal{J}\Gamma(G)} F(H)\Big)^G.$$

3.3 Examples. (1) Let $\mathcal{A}(G)$ be the full subcategory of $\mathcal{S}(G)$ with the abelian subgroups as objects. Then, for any n,

$$C_{\mathbf{Z}^n,\mathsf{F}}(G) \simeq \lim_{\mathcal{A}(G)} C_{\mathbf{Z}^n,\mathsf{F}}(A).$$

(2) Let $\mathcal{E}_p(G)$ be the full subcategory of $\mathcal{S}(G)$ with the elementary abelian p-groups as objects. Then, if V is an F_p-vector space,

$$C_{V,\mathsf{F}}(G) \simeq \lim_{\mathcal{E}_p(G)} C_{V,\mathsf{F}}(E).$$

(3) Let $\mathcal{C}(G)$ be the full subcategory of $\mathcal{S}(G)$ with the cyclic groups as objects. Then

$$C_{\mathbf{Z},\mathsf{F}}(G) \simeq \lim_{\mathcal{C}(G)} C_{\mathbf{Z},\mathsf{F}}(C).$$

Our little observation can be utilized in another way. Note that there is a natural isomorphism of G-sets

$$\coprod_Q \mathrm{Epi}(\Gamma, Q) \times_{\mathrm{Aut}(Q)} \mathrm{Mono}(Q, G) = \mathrm{Hom}(\Gamma, G),$$

where the union is over isomorphism classes of finite groups. For any fixed G, this is clearly just a finite disjoint union. The group $\mathrm{Aut}(Q)$ acts freely on both $\mathrm{Epi}(\Gamma, Q)$ and $\mathrm{Mono}(Q, G)$. Furthermore, $\mathrm{Map}_G(\mathrm{Mono}(Q, G), \mathsf{F})$ is easily seen to be a quotient Mackey functor of $C_{Q,\mathsf{F}}(G)$. These observations imply

3.4 Proposition.

$$C_{\Gamma,\mathsf{F}}(G) \simeq \prod_Q |\mathrm{Epi}(\Gamma, Q)| \times \frac{1}{|\mathrm{Aut}(Q)|} \times \mathrm{Map}_G(\mathrm{Mono}(Q, G), \mathsf{F}),$$

as Mackey functors. (Here $n \times R$ means the n-fold product $R \times R \times \cdots \times R$.)

To see how this proposition can be used, we generalize counting arguments that occurred in [K2] and [HKR]. Let G be a finite group. The group ring $\mathsf{F}_p[\mathrm{Out}(G)]$ acts on $C_{\Gamma,\mathsf{F}_p}(G)$ as does the larger F_p-algebra $A(G, G) \otimes \mathsf{F}_p$ generated by inductions composed with homomorphisms. If G is a p-group, this latter algebra is known to be isomorphic to the stable endomorphism ring $\{BG_+, BG_+\} \otimes \mathsf{F}_p$ (as in [Ma]). Let e be an idempotent in either $\mathsf{F}_p[\mathrm{Out}(G)]$ or $A(G, G) \otimes \mathsf{F}_p$. Let $f_e(n) = \dim_{\mathsf{F}_p} eC_{\mathbf{Z}_p^n,\mathsf{F}_p}(G)$. In Section 5, it will be shown that this function of n has topological meaning: $f_e(n)$ is the nth Morava K-theory Euler characteristic of the spectrum eBG_+. Note that $f_1(n) = |\mathrm{Hom}(\mathbf{Z}_p^n, G)/G|$.

Our application of Proposition 3.4 is

3.5 Proposition. *For all p-groups G and e, $f_e(n)$ is a polynomial in p^n, with rational coefficients, of degree $\leq d$ where p^d is the order of the largest abelian subgroup of G.*

To prove this, if Q is an abelian p-group, let $f_Q(n) = |\mathrm{Epi}(\mathbf{Z}^n, Q)|$. By Proposition 3.4,

$$f_e(n) = \sum_Q \frac{1}{|\mathrm{Aut}(Q)|} \times \dim_{\mathsf{F}_p} e\mathrm{Map}_G(\mathrm{Mono}(Q, G), \mathsf{F}_p) \times f_Q(n),$$

so that $f_e(n)$ is a linear combination of functions $f_Q(n)$. It suffices to verify the next lemma.

3.6 Lemma. *If $|Q| = p^d$, $f_Q(n)$ is a polynomial in p^n of degree d.*

Proof. Observe that

$$\mathrm{Hom}(\mathbf{Z}_p^n, Q) = \coprod_{Q' < Q} \mathrm{Epi}(\mathbf{Z}_p^n, Q').$$

Noting that $|\mathrm{Hom}(\mathbf{Z}_p^n, Q)| = |Q^n| = p^{nd}$, this implies that

$$f_Q(n) = p^{nd} - \sum_{Q' \lneq Q} f_{Q'}(n).$$

The result follows by induction on $|Q|$.

3.7 Remarks. (i) The denominators in the rational coefficients occurring in $f_e(n)$ are just due to the factors $1/|\mathrm{Aut}(Q)|$.

(ii) For related, but slightly different, counting arguments, see [HKR, §3].

(iii) For some examples of $f_e(n)$'s when G is abelian, see [K2, §6].

4 H*(BG) revisited

In this section, $H^*(G)$ denotes $H^*(BG; \mathbf{F}_p)$, and V denotes a finite-dimensional $\mathbf{F}_p$-vector space. Let $\mathcal{U}$ be the category of unstable modules over the Steenrod algebra, and let $\mathcal{K}$ be the category of unstable A-algebras. Lannes and Schwartz [HLS] have re-examined Quillen's work on $H^*(G)$ using unstable A-module technology. We place their results in our character ring context.

There is a natural map

$$\mathbf{F}_p[\mathrm{Hom}(V,G)] \longrightarrow \mathrm{Hom}_{\mathcal{U}}(H^*(G), H^*(V)),$$

where $\mathbf{F}_p[\mathrm{Hom}(V,G)]$ is an $\mathbf{F}_p$-vector space with basis $\mathrm{Hom}(V,G)$. Since inner automorphisms of G induce the identity in cohomology, this factors through

$$\mathbf{F}_p[\mathrm{Hom}(V,G)/G] \longrightarrow \mathrm{Hom}_{\mathcal{U}}(H^*(G), H^*(V)).$$

Taking (profinite) duals yields a natural map

$$\chi_G \colon \mathrm{Hom}_{\mathcal{U}}(H^*(G), H^*(V))^* \longrightarrow C_{V,\mathbf{F}_p}(G).$$

4.1 Theorem [HLS]. *χ_G is an isomorphism of Mackey functors. Furthermore, taking Spec of both rings yields a natural bijection*

$$\mathrm{Hom}(V,G)/G \longrightarrow \mathrm{Hom}_{\mathcal{K}}(H^*(G), H^*(V)).$$

We sketch the proof. Firstly, it is straightforward to check that χ_G commutes with induction maps — using the double coset formula, one is reduced to the following well-known lemma.

4.2 Lemma. *If V' is a proper subspace of the $\mathbf{F}_p$-vector space V, the cohomology transfer $\mathrm{ind}_{V'}^{V} \colon H^*(V') \to H^*(V)$ is the zero map.*

The proof that χ_G is an isomorphism then uses four deep facts:

(1) [Q1] $\alpha_G\colon H^*(G) \to \lim_{\mathcal{E}_p(G)} H^*(E)$ is an F-isomorphism;

(2) [LS] a map $M \to N$ in $\mathcal{K}$ is an F-isomorphism if and only if the induced map $\mathrm{Hom}_{\mathcal{U}}(N, H^*(V)) \to H^*_{\mathcal{U}}(M, H^*(V))$ is an isomorphism for all V;

(3) [C1,Mi] $H^*(V)$ is injective in $\mathcal{U}$; and

(4) [AGM] χ_E is an isomorphism if E is elementary abelian.

Armed with these, it is easy to show that χ_G is an isomorphism. Consider the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{U}}(H^*(G), H^*(V))^* & \xrightarrow{\chi_G} & C_{V,\mathbf{F}_p}(G) \\
\downarrow \alpha & & \\
\mathrm{Hom}_{\mathcal{U}}(\lim_{\mathcal{E}_p(G)} H^*(E), H^*(V))^* & & \Big\downarrow \gamma \\
\downarrow \beta & & \\
\lim_{\mathcal{E}_p(G)} \mathrm{Hom}_{\mathcal{U}}(H^*(E), H^*(V))^* & \xrightarrow{\lim \chi_E} & \lim_{\mathcal{E}_p(G)} C_{V,\mathbf{F}_p}(E).
\end{array}$$

The map α is an isomorphism by facts (1) and (2), β is an isomorphism by (3), and (4) implies that $\lim \chi_E$ is an isomorphism. In Example 3.3 (2), we showed that γ is an isomorphism; thus, χ_G is also.

Of course, χ_G is functorial in V as well as G. This is the functoriality exploited by Lannes and Schwartz when they assign an "analytic functor" to an unstable A-module [HLS]. This leads to a rather practical group-theoretic way to prove that $H^*(G)$ has many nilpotent elements.

If S is a simple $\mathbf{F}_p[\mathrm{Out}(G)]$ or $A(G,G) \otimes \mathbf{F}_p$ module, let e_S be an associated idempotent, so that $e_S M = 0$ exactly when S does not occur as a composition factor in M.

4.3 Theorem. *Let G be a p-group of rank d, and let e be an idempotent in either $\mathbf{F}_p[\mathrm{Out}(G)]$ or $A(G,G) \otimes \mathbf{F}_p$. Then*

$$e\, C_{(\mathbf{Z}/p)^d, \mathbf{F}_p}(G) = 0 \iff e H^*(G) \subset \ker(\alpha_G),$$

where $\alpha_G\colon H^*(G) \to \lim_{\mathcal{E}_p(G)} H^*(E)$ is the natural map. In particular, if S is not a composition factor in $C_{(\mathbf{Z}/p)^d, \mathbf{F}_p}(G)$, then $e_S H^*(G)$ is all nilpotent.

Before proving this we look at some examples.

4.4 Example. Let $G = Q_8$, the quaternion group of order 8. $\mathrm{Out}(Q_8) \simeq \Sigma_3$, permutations of i, j, k. The $\mathrm{Out}(Q_8)$-set $\mathrm{Hom}(\mathbf{Z}/2, Q_8)$ has trivial action, since $Z(Q_8) = \{\pm 1\}$ is the only nontrivial elementary abelian subgroup. However, $\mathbf{F}_2[\mathrm{Out}(Q_8)]$ has *two* simple modules, the trivial module and a two-dimensional one. We conclude that $H^*(Q_8)$ decomposes (over the Steenrod algebra) as $H^*(Q_8) = M_0 \oplus M_1 \oplus M_1$, where $M_1 \subset \ker(\alpha_{Q_8})$. We note that $M_0 \simeq H^*(SL_2(\mathbf{F}_3))$, and $M_1 \simeq H^*(\Sigma^{-1} BS^3/BN)$ [MP].

4.5 Example. If $\alpha_G: H^*(G) \to \lim\limits_{\mathcal{E}_p(G)} H^*(E)$ is monic, it follows that every simple $\mathrm{Out}(G)$ (and $A(G,G)\otimes\mathbf{F}_p$) module occurs as a composition factor in $C_{V,\mathbf{F}_p}(G)$ with $V \geq \operatorname{rank} G$. In [Q2], it is shown that if α_G is monic, then so is $\alpha_{\mathbf{Z}/p \wr G}$. In particular, if G is the p-Sylow subgroup of a symmetric group, then the map α_G is monic.

We sketch the proof of Theorem 4.3. Reasoning as in Section 3 (e.g., Proposition 3.4), shows that

$$e\, C_{(\mathbf{Z}/p)^d,\mathbf{F}_p}(G) = 0 \iff \forall V,\ e\, C_{V,\mathbf{F}_p}(G) = 0.$$

Let $\mathcal{H}^*(G) = \lim\limits_{\mathcal{E}_p(G)} H^*(E)$. In the language of Lannes and Schwartz, $\mathcal{H}^*(G)$ is "nil-closed", and since α_G is an F-isomorphism, $\mathcal{H}^*(G)$ is the nil-closure of $H^*(G)$. Lannes and Schwartz show that a nil-closed object M is *equivalent* to the functor $V \mapsto \operatorname{Hom}_{\mathcal{U}}(M, H^*(V))^*$. Thus $\mathcal{H}^*(G)$ is equivalent to $V \mapsto C_{V,\mathbf{F}_p}(G)$. Thus, if e is an idempotent,

$$\begin{aligned} e\, C_{V,\mathbf{F}_p}(G) = 0 \text{ for all } V &\iff e\,\mathcal{H}^*(G) = 0 \\ &\iff e\, H^*(G) \subset \ker \alpha_G. \end{aligned}$$

4.6 Remark. If $e \neq 0$, then $e\,H^*(G)$ is always infinite dimensional, a consequence of the fact that $\{BG_+, BG_+\}$ is torsion free: if $e\,H^*(G)$ were finite dimensional, then the order of the identity in $\{eBG_+, eBG_+\}$ would be finite.

5 Characters for complex-oriented theories

We first summarize some theorems that will appear in [HKR]. As motivation, recall that

$$\chi_G: R(G) \longrightarrow C_{\mathbf{Z},\mathbf{C}}(G)$$

is an embedding of a maximal $\mathbf{Z}$-lattice, and that

$$C_{\mathbf{Z},\mathbf{C}}(G) \simeq \lim_{\mathcal{C}(G)} C_{\mathbf{Z},\mathbf{C}}(C).$$

This suggests that perhaps $R(G) \simeq \lim\limits_{\mathcal{C}(G)} R(C)$. This is almost true: Artin's Theorem [S] says that there is an isomorphism

$$R(G) \otimes \mathbf{Z}[1/|G|] \simeq \lim_{\mathcal{C}(G)} R(C) \otimes \mathbf{Z}[1/|G|].$$

It turns out that the splitting principle and equivariant general nonsense suffice to prove an analogous theorem.

5.1 Theorem [HKR]. *Let E^* be any complex-oriented theory. For any finite group G, the natural map*

$$E^*(BG) \otimes \mathbf{Z}[1/|G|] \longrightarrow \lim_{\mathcal{A}(G)} E^*(BA) \otimes \mathbf{Z}[1/|G|]$$

is an isomorphism.

This suggests that character rings for detecting complex-oriented theories should be of the form $C_{\Gamma,\mathsf{F}}(G)$ with Γ abelian, so that $\Gamma(G) \subset \mathcal{A}(G)$, and Proposition 3.2 applies. Our main discovery is that this can be done, with the rank of Γ corresponding to the type of v_n-periodicity present in E^*.

To describe one version of our result, we recall some notation. Localized at a prime p, MU is equivalent to a wedge of suspensions of the Brown-Peterson spectrum BP. The coefficients are $BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ with $|v_i| = 2p^i - 2$. A complex-oriented theory E^* is *v_n-periodic* if v_n is a unit in E^*. For example, K-theory is v_1-periodic. Let $I_n \subset BP_*$ be the ideal $(p, v_1, v_2, \ldots, v_{n-1})$. Finally, recall that to a complex-oriented theory E^*, there is an associated formal group law F, and $[m](x) \in E^*[[x]]$ denotes the m-fold formal sum $x +_F x +_F \cdots +_F x$.

5.2 Theorem [HKR]. *Let E^* be a multiplicative v_n-periodic cohomology theory such that the coefficient ring is a characteristic 0 domain and is complete in the I_n-adic topology. Let $\mathsf{F}(E^*)$ be the graded field of fractions of E^* with solutions to $[p^i](x) = 0$ adjoined, for all i, as elements in degree 2. Then there is an isomorphism of Mackey functors*

$$\chi_G : E^*(BG) \otimes_{E^*} \mathsf{F}(E^*) \simeq C_{\mathbf{Z}_p^n, \mathsf{F}(E^*)}(G).$$

We sketch the proof, emphasizing how it parallels the proof of the $H^*(G)$ calculation of Section 4. Firstly, χ_G is again constructed using

$$\mathrm{Hom}(\mathbf{Z}_p^n, G)/G \longrightarrow \mathrm{Hom}_{E^*}(E^*(BG), E^*(B\mathbf{Z}_p^n)),$$

where by $E^*(B\mathbf{Z}_p^n)$ we mean $\mathrm{colim}_N\, E^*(B(\mathbf{Z}/p^N)^n)$. Adjointing yields a natural map

$$E^*(BG) \longrightarrow C_{\mathbf{Z}_p^n, E^*(B\mathbf{Z}_p^n)}(G).$$

Formal group law theory yields a map of E^*-algebras

$$\theta : E^*(B\mathbf{Z}_p^n) \longrightarrow \mathsf{F}(E^*)$$

which, when composed with the above, defines

$$\chi_G : E^*(BG) \longrightarrow C_{\mathbf{Z}_p^n, \mathsf{F}(E^*)}(G).$$

To check that χ_G commutes with induction maps, we use an analogue of Lemma 4.2.

5.3 Lemma [HKR]. *If A' is a proper subgroup of an abelian p-group A, the composite*

$$E^*(BA') \xrightarrow{\text{ind}} E^*(BA) \xrightarrow{\alpha^*} E^*(B\mathbf{Z}_p^n) \xrightarrow{\theta} \mathsf{F}(E^*)$$

is zero, for all surjective homomorphisms $\alpha\colon \mathbf{Z}_p^n \to A$.

The proof that χ_G is an isomorphism (after extending scalars) then goes as before. The map χ_A is explicitly seen to be an isomorphism if A is abelian — this is the fact corresponding to the Adams-Gunawardena-Miller Theorem. Our generalized Artin's theorem then takes the place of Quillen's F-isomorphism theorem, and thus χ_G can be identified with $\lim\limits_{\mathcal{A}(G)} \chi_A$.

We now use Theorem 5.2 to obtain a slight strengthening of another result in [HKR]:

$$\dim_{K(n)^*} K(n)^{\text{even}}(BG) - \dim_{K(n)^*} K(n)^{\text{odd}}(BG) = |\text{Hom}(\mathbf{Z}_p^n, G)/G|.$$

The strengthening goes as follows. Note that both $G \mapsto K(n)^{\text{even}}(BG)$ and $G \mapsto K(n)^{\text{odd}}(BG)$ are Mackey functors to the category of $K(n)^*$-modules. So one can view $K(n)^{\text{even}}(BG) - K(n)^{\text{odd}}(BG)$ as a "virtual Mackey functor". Call this $\chi_n(G)$

5.3 Theorem. *$[\chi_n(G)] = [C_{\mathbf{Z}_p^n, K(n)^*}(G)]$ in the Grothendieck ring of virtual Mackey functors.*

By this statement, we just mean that for all G, $[\chi_n(G)] = [C_{\mathbf{Z}_p^n, K(n)^*}(G)]$ as virtual representations of the algebra $A(G,G) \otimes \mathbf{Z}/p$ (and so also as virtual $\mathsf{F}_p[\text{Out}(G)]$-modules).

To prove Theorem 5.3, we apply the previous theorem to a specific theory. Using the Bass-Sullivan construction, one can construct a theory E^* with coefficients $\mathbf{Z}_p[v_n, v_n^{-1}]$. The theory E^* can be given a product [Mo][(1)]. Furthermore, there will be a cofibration sequence

$$E \xrightarrow{p} E \longrightarrow K(n),$$

and thus an associated Bockstein spectral sequence.

By Theorems 5.1 and 5.2, $E^*(BG)/(\text{torsion})$ embeds in $C_{\mathbf{Z}_p^n, \mathsf{F}(E^*)}(G)$ as a maximal E^*-lattice. Another such lattice is given by $C_{\mathbf{Z}_p^n, E^*}(G)$. By standard arguments in the theory of modular representations (as in [S, page 125]), one can conclude that

$$[E^*(BG)/(\text{torsion}) \otimes_{E^*} K(n)^*] = [C_{\mathbf{Z}_p^n, E^*}(G) \otimes_{E^*} K(n)^*]$$

as virtual Mackey functors. Note that

$$C_{\mathbf{Z}_p^n, E^*}(G) \otimes_{E^*} K(n)^* = C_{\mathbf{Z}_p^n, K(n)^*}(G).$$

(1) I would like to thank J. Morava for assuring me that this is still true when $p = 2$.

Now we use the Bockstein spectral sequence, with $E^1 = K(n)^*(BG)$, and $E^\infty = E^*(BG)/(\text{torsion}) \otimes_{E^*} K(n)^*$, the latter all in even degrees. Homology preserves Euler characteristics, so $\chi(E^r) = \chi(E^{r+1})$, for all r, as virtual Mackey functors. We conclude that

$$[\chi_n(G)] = [\chi(E^1)] = [\chi(E^\infty)] = [C_{\mathbf{Z}_p^n, K(n)^*}(G)],$$

as needed.

Our analogue of Theorem 4.3 is the following.

5.4 Theorem. *Let G be a p-group of rank d, and let e be an idempotent element in either $\mathbf{F}_p[\mathrm{Out}(G)]$ or $A(G,G) \otimes \mathbf{F}_p$. Then $e\,C_{\mathbf{Z}_p^n, \mathbf{F}_p}(G) = 0$ implies that $e\,K(n)^{\mathrm{odd}}(BG) \neq 0$ for some n. In particular, if there exists a simple $\mathbf{F}_p[\mathrm{Out}(G)]$ or $A(G,G) \otimes \mathbf{F}_p$ module* not *occurring as a composition factor in $C_{\mathbf{Z}_p^n, \mathbf{F}_p}(G)$, then $K(n)^*(BG)$ is not concentrated in even degrees*[1].

Proof. By the last theorem, $e\,C_{\mathbf{Z}_p^n, \mathbf{F}_p}(G) = 0$ implies that for all n, $\dim eK(n)^{\mathrm{even}}(BG) - \dim eK(n)^{\mathrm{odd}}(BG) = 0$. If $e\,K(n)^{\mathrm{odd}}(BG) = 0$, we would have $0 = e\,K(n)^*(BG) = K(n)^*(eBG)$, for all n. Here eBG denotes the stable retract of $\Sigma^\infty BG_+$ split off by e. We claim that this is impossible, i.e., *no* retract of $\Sigma^\infty BG_+$ can be $K(n)^*$-acyclic for all n. To see this we use the concept of "harmonic spectra" from [R]. By definition, a spectrum Y is harmonic if $[X,Y] = 0$ for all spectra X such that $K(n)_*(X) = 0$ for all n. Any retract of a harmonic spectrum will be harmonic, and if Y is harmonic and not contractible, it follows that $K(n)^*(Y) \neq 0$ for some n. Our proof is completed by the next lemma, due to Mike Hopkins.

5.5 Lemma. *$\Sigma^\infty BG_+$ is harmonic, for all finite groups G.*

Proof. By transfer arguments, we can assume that G is a p-group. In [R], it is shown that any finite complex is harmonic. It is formal that if Y is harmonic, so is any function spectrum $F(Z,Y)$. Thus Spanier-Whitehead duals, $F(Z,S^0)$'s are harmonic. By the Segal conjecture [C2], $\Sigma^\infty BG_+$ is a retract of its own dual.

Now, of course, we wish to find a group G satisfying the criterion of Theorem 5.4, analogous to Example 4.4. We have been unable to find one. The next proposition suggests where one should look for a G.

5.6 Proposition. *Suppose G is a p-group having a simple $\mathbf{F}_p[\mathrm{Aut}(G)]$ module not appearing in $\mathrm{Map}(\mathrm{Hom}(\mathbf{Z}_p^n, G), \mathbf{F}_p)$ for all n. Then G contains a characteristic "class two" subgroup with the same property.*

By class two, we mean a group H such that $H/Z(H)$ is abelian.

[1] (added in proof) John Thompson has shown me that *no* such $\mathrm{Out}(G)$-module can exist.

Proof. Our hypothesis on G is equivalent to the existence of a nonzero idempotent $e \in \mathbb{F}_p[\mathrm{Aut}(G)]$ such that $e\,\mathrm{Map}(\mathrm{Hom}(\mathbb{Z}_p^n, G), \mathbb{F}_p) = 0$ for all n. In the famous Feit-Thompson odd order paper, J. Thompson proved that every finite p-group G contains a characteristic class two subgroup H such that all p'-elements of $\mathrm{Aut}(G)$ act nontrivially on H [G, page 185]. It follows that there are no idempotents in the kernel of $\mathbb{F}_p[\mathrm{Aut}(G)] \to \mathbb{F}_p[\mathrm{Aut}(H)]$, in particular, e projects to a nonzero element of $\mathbb{F}_p[\mathrm{Aut}(H)]$. Since $\mathrm{Map}(\mathrm{Hom}(\mathbb{Z}_p^n, G), \mathbb{F}_p) \to \mathrm{Map}(\mathrm{Hom}(\mathbb{Z}_p^n, H), \mathbb{F}_p)$ is surjective, $e\,\mathrm{Map}(\mathrm{Hom}(\mathbb{Z}_p^n, H), \mathbb{F}_p) = 0$ for all n, also.

5.7 Remark. In fact, H can be chosen so that $H/Z(H)$ is elementary abelian. It is unclear whether $Z(H)$ can also be taken to be elementary.

6 Equivariant cohomology theories

A theme familiar to topologists is introduced in this section. Our character rings are trying to be the coefficients of equivariant cohomology theories.

We need the following construction.

6.1 Definition. Let Γ and G be topological groups, with G compact Lie. Define functors

$$F_\Gamma(G;\bullet): G\text{-CW-complexes} \to G\text{-CW-complexes}$$

by $F_\Gamma(G;X) = \{(\alpha, x) \in \mathrm{Hom}(\Gamma, G) \times X \mid \alpha(\gamma)x = x \text{ for all } \gamma \in \Gamma\}$. This is a sub G-space of $\mathrm{Hom}(\Gamma, G) \times X$, where G acts diagonally on the product. Note that if $\mathrm{Hom}(\Gamma, G)$ is finite, then

$$F_\Gamma(G;X) = \coprod_{\alpha \in \mathrm{Hom}(\Gamma, G)} X^{\mathrm{Im}(\alpha)}.$$

We list some basic properties of these functors.

6.2 Proposition.

(i) $F_\Gamma(G;*) = \mathrm{Hom}(\Gamma, G)$.

(ii) $F_\Gamma(G;\bullet)$ *preserves* G*-pushouts.*

(iii) *If* $H < G$, *and* X *is an* H*-space, there is a natural* G*-homeomorphism*

$$G \times_H F_\Gamma(H;X) \simeq F_\Gamma(G; G \times_H X).$$

(iv) *There is a natural* G*-homeomorphism*

$$\operatorname*{colim}_{\mathcal{J}\Gamma(G)} F_\Gamma(H;X) \simeq F_\Gamma(G;X).$$

(v) $F_\Gamma(G; F_\Lambda(G;X)) = F_{\Gamma\times\Lambda}(G;X)$.

All of these can be verified in a straightforward manner. Checking (v) is a recommended exercise.

Properties (i)–(iii) imply the following, letting G run through finite groups.

6.3 Corollary. *Let E^* be a (non-equivariant) cohomology theory, and F a characteristic zero field. The assignment*

$$X \mapsto E^*(F_\Gamma(G;X))^G \otimes \mathsf{F}.$$

is an equivariant cohomology theory defined for finite G-CW-complexes X.

Since $\mathrm{Hom}(\Gamma, G)$ is discrete, the coefficients $E^*(\mathrm{Hom}(\Gamma, G))^G \otimes \mathsf{F}$ can be identified with $C_{\Gamma, E^*\otimes\mathsf{F}}(G)$. Furthermore, it is easy to extend the formula for induction.

As an example of how one uses such a theory, we prove

6.4 Theorem. *If G is a finite group, and X is a finite G-CW-complex, there is a natural isomorphism*

$$\chi_G: K_G(X) \otimes \mathsf{C} \longrightarrow K(F_{\mathsf{Z}}(G;X))^G \otimes \mathsf{C}.$$

Proof. We need to define χ_G. A homomorphism $\alpha: \mathsf{Z} \to G$ factors as $\mathsf{Z} \twoheadrightarrow \mathsf{Z}/n \rightarrowtail G$ for some n. The inclusion of Z/n-spaces $X^{\mathsf{Z}/n} \subset X$ induces $K_G(X) \to K_{\mathsf{Z}/n}(X^{\mathsf{Z}/n})$. Now recall that $K_H(Y) = K(Y) \otimes R(H)$ if Y is a trivial H-space [Seg] and that $R(\mathsf{Z}/n) = \mathsf{Z}[x]/(x^n - 1)$. The component of χ_G landing in $K(X^{\mathrm{Im}(\alpha)}) \otimes \mathsf{C}$ is the composite

$$\begin{aligned} K_G(X) \longrightarrow K_{\mathsf{Z}/n}(X^{\mathrm{Im}(\alpha)}) = K(X^{\mathrm{Im}(\alpha)}) \otimes \mathsf{Z}[x]/(x^n - 1) \\ \longrightarrow K(X^{\mathrm{Im}(\alpha)}) \otimes \mathsf{C}, \end{aligned}$$

where the last map is defined by sending x to $e^{2\pi i/n}$.

The proof that χ_G is an isomorphism is immediate. When $X = *$, χ_G reduces to the usual character map

$$R(G) \otimes \mathsf{C} \longrightarrow C_{\mathsf{Z},\mathsf{C}}(G).$$

This is an isomorphism, and one inducts on the cells of X.

6.5 Examples. (1) If $G = \mathsf{Z}/3$, the theorem says that

$$K_{\mathsf{Z}/3}(X) \otimes \mathsf{C} = [K(X) \oplus K(X^{\mathsf{Z}/3}) \oplus K(X^{\mathsf{Z}/3})] \otimes \mathsf{C}.$$

(2) Note that by 6.2(v), $F_{\mathsf{Z},\mathsf{C}}(\mathrm{Hom}(\mathsf{Z}^{n-1}, G)) = \mathrm{Hom}(\mathsf{Z}^n, G)$. Thus the ring $K_G(\mathrm{Hom}(\mathsf{Z}^{n-1}, G))$ is detected by $C_{\mathsf{Z}^n,\mathsf{C}}(G)$ (or, more generally, $C_{\mathsf{Z}^n,\mathsf{F}}(G)$, where F is any char 0 field that contains $|G|$th roots of 1). If G is a p-group, the Atiyah-Segal theorem [AS] implies that $K_G(\mathrm{Hom}(\mathsf{Z}^{n-1}, G))_p = K(EG \times_G \mathrm{Hom}(\mathsf{Z}^{n-1}, G))$.

We leave it to the reader to check that the fact that $EG \times_G G_{\mathrm{conj}} \simeq BG^{S^1}$ (see e.g., [DZ]) implies that $EG \times_G \mathrm{Hom}(\mathbb{Z}^{n-1}, G) \simeq BG^{T^{n-1}}$, when G is finite. Thus $K^*(BG^{T^{n-1}})$ is detected by $C_{\mathbb{Z}^n, \overline{\mathbb{Q}}_p}(G)$, the same ring that detects $E^*(BG)$ when E^* is v_n-periodic. However, in the simple example $n = 2$ and $G = \mathbb{Z}/p$, the ring structures of $K((B\mathbb{Z}/p)^{S^1})$ and $E^*(B\mathbb{Z}/p)$ are different: the former looks roughly like an associated graded of the latter.

7 Simplicial character rings

It has occurred to a number of people who have heard about our character rings for v_n-periodic theories that they should be assembled into some sort of co-simplicial object. Here I wish to advertise a question motivated by this idea.

Recall that BG is the geometric realization of the simplicial set $\underline{n} \mapsto G^n = \mathrm{Hom}(F_n, G)$. This suggests the following constructions.

7.1 Definitions. *Let Γ be a co-simplicial group.*

(i) Let $B_\Gamma G$ be the realization of $\underline{n} \mapsto Hom(\Gamma_n, G)$.

(ii) Let $\overline{B}_\Gamma G$ be the realization of $\underline{n} \mapsto Hom(\Gamma_n, G)/G$.

Note that the chain complex computing $H^*(\overline{B}_\Gamma G; \mathsf{F})$ has n-chains isomorphic to $C_{\Gamma_n, \mathsf{F}}(G)$. This looks promising. However, this construction is too naive; induction does not generally commute with our differentials. Still, it is tempting to think that $\overline{B}_{(\mathbb{Z}/p)^\bullet} G$ has something to do with $H^*(G; \mathbb{Z}/p)$ and that $\overline{B}_{\mathbb{Z}^\bullet} G$ has something to do with v_n-periodicity.

7.2 Problem. Find a functor

$$\mathcal{B} : \text{finite groups} \longrightarrow \text{simplicial sets}$$

such that

(i) there is a natural homotopy equivalence $\Sigma^\infty |\mathcal{B}G| \simeq \Sigma^\infty BG$, and

(ii) the chain complex $C^*(\mathcal{B}G; \mathsf{F})$ is a complex of Mackey functors inducing the transfer in cohomology.

The point here is that when one defines the usual stable homotopy transfer for $H < G$, $\Sigma^\infty BG_+ \to \Sigma^\infty BH_+$, one "trades in" BH for $EG \times_G (G/H)$, a homotopic, but different, space.

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